Flat manifolds, harmonic spinors, and eta invariants

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Abstract

The aim of this paper is to calculate the eta invariants and the dimensions of the spaces of harmonic spinors of an infinite family of closed flat manifolds $\mathcal{F}_{CHD}$. It consists of some flat manifolds M with cyclic holonomy groups. If $M \in \mathcal{F}_{CHD}$, then we give explicit formulas for $\eta(M)$ and $\mathfrak{h}(M)$. The are expressed in terms of solutions of appropriate congruences in $\{-1,1\}^{\left[\frac{n-1}{2}\right]}$. As an application we investigate the integrability of some η invariants of $\mathcal{F}_{CHD}$-manifolds.

Key words and phrases: *Spin structure, harmonic spinor, eta invariant, flat manifold.*

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1 Introduction

In this paper we consider Dirac operators on an infinite family $\mathcal{F}_{CHD}$ of closed flat manifolds. It consists of flat manifolds M with cyclic holonomy groups of odd order equal to the dimension of M. The family $\mathcal{F}_{CHD}$ is particularly simple and the investigation of different properties of multidimensional flat manifolds should start with the investigation of them in this particular case. Some $\mathcal{F}_{CHD}$ manifolds arises in the classification of flat manifolds whose holonomy groups have prime order (cf. [4]). We describe the eta invariants

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of the Dirac operators arising from different spin structures and we give necessary and sufficient conditions of the existence of nontrivial harmonic spinors. The methods used here extends that used in [12]. We apply them to much wider class of manifolds and we consider related general questions.

To formulate the main results we need some definitions. Let $n = 2k+1$ be an odd number, and let $a_1, ..., a_n$ be a basis of $\mathbb{R}^n$. Consider the linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A(a_j) = a_{j+1}$ for $j < n-1$, $A(a_{n-1}) = -a_1 - ... - a_{n-1}$, and $A(a_n) = a_n$. Let $a = \frac{1}{n} a_n$ and let $g(x) = A(x) + a$. An n-dimensional flat manifold $M \in \mathcal{F}_{CHD}$ can be written as $\mathbb{R}^n/\Gamma$, where $\Gamma = \langle a_1, ..., a_{n-1}, g\rangle$. The linear part A of g has two lifts $\alpha_+, \alpha_- \in Spin(n)$ such that $\alpha_+^n = id$ and $\alpha_-^n = -id$ (see Section 2). This defines two spin structures on M.

To formulate the result describing $\eta_{M^n}(0)$ for $M^n \in \mathcal{F}_{CHD}$ we need some combinatorial invariants. It is known that $\eta_{M^n}(0) = 0$ if k is even (cf. [1, p. 61]) so we consider the case when k is odd. Let

$$c(k) = \begin{cases} 0 & \text{if } \frac{k(k+1)}{2} \text{ is even} \\ \frac{1}{2} & \text{if } \frac{k(k+1)}{2} \text{ is odd} \end{cases} .$$

For every $\epsilon = (\epsilon_1, ..., \epsilon_k) \in \{-1, 1\}^k$ consider

$$\mu_\epsilon = \sum_{j=1}^{k} \epsilon_j j \quad \text{and} \quad \nu(\epsilon) = \epsilon_1 \cdots \epsilon_k.$$

Let $\mathcal{D}_+ = \{\epsilon \in \{-1, 1\}^k : \nu(\epsilon) = \epsilon_1\epsilon_2...\epsilon_k = 1\}$, let $r \in \{0, ..., n-1\}$, let

$$A_r^+ = 2\#\{\epsilon \in \mathcal{D}_+ : \frac{\mu_\epsilon}{2} + c(k)n \equiv r \bmod (n)\}$$

in the case of α_+, and let

$$A_r^- = 2\#\{\epsilon \in \mathcal{D}_+ : \frac{\mu_\epsilon}{2} + c(k)n + k \equiv r \bmod (n)\}$$

in the case of α_-. The numbers $A_r^\pm$ are well defined (cf. Remark 1).

Theorem 1. *Let k be an odd positive integer and let $n = 2k + 1$. If $M^n \in \mathcal{F}_{CHD}$, then*

$$\eta_{M^n,\alpha_+}(0) = \sum_{r=1}^{n-1} A_r^+ \left(1 - \frac{2r}{n}\right), \tag{1}$$

$$
(2) \qquad \eta_{M^n,\alpha_-}(0) = \sum_{r=0}^{n-1} A_r^- \left(1 - \frac{2r+1}{n}\right).
$$

Applying Theorem 1 we prove that some η-invarints of $\mathcal{F}_{CHD}$-manifolds are integral (Corollary 1) and that $\eta_{M,\alpha_+} - \eta_{M,\alpha_-} \in 2\mathbb{Z}$ (Corollary 2). Let $\mathfrak{h}(V)$ be the dimension of the vector space of harmonic spinors.

Now we state another result of the paper.

Proposition 1. *Let k be a positive integer and let $n = 2k+1$. If $M \in \mathcal{F}_{CHD}$, then*

a) $\mathfrak{h}(M, \alpha_+) > 0$ *if and only if* $n \geq 5$.

b) $\mathfrak{h}(M, \alpha_-) = 0$.

The spectra of the Dirac operators on flat tori were described in [6]. The spectra of the Dirac operators on closed 3-dimensional flat manifolds and their eta invariants were calculated in [12]. We should also mention about ([11]) where the authors consider spin structure and the Dirac operators on flat manifolds with $\mathbb{Z}_p$, (p-prime number), and non-cyclic holonomy.

Throughout this paper the following notation will be used. If G is a group and $g_1, ..., g_l \in G$, then $\langle g_1, ..., g_l \rangle$ is the subgroup of G generated by $g_1, ..., g_l$. The symbol X^G stands for the set of the fixed points of an action of G on X. For every $g \in G$, $X^g = \{x \in X : gx = x\}$. By Γ (or Γ_n) we denote the deck group of a closed flat manifold M, by h the holonomy homomorphism of M, and by $\widehat{h}$ its lift to $Spin(n)$. The standard epimorphism from $Spin(n)$ to $SO(n)$ will be denoted by λ (cf. Section 2). The letter Γ_0 stands for the maximal abelian subgroup of Γ consisting of all translations belonging to Γ, (cf. [4] and [15]). By $a_1, ..., a_n$ we usually denote a basis of Γ_0. The subspace of a vector space spanned by vectors $v_1, ..., v_l$ will be denoted by $\mathrm{Span}[v_1, ..., v_l]$. The symbols α_+, α_-, $\mathfrak{h}(M)$, $c(k)$, μ_ϵ, $\nu(\epsilon)$, and A_r were defined above. The cyclic group $\langle A \rangle$ will be denoted by G.

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2 Spin structures on $\mathcal{F}_{CHD}$-manifolds and Dirac operators

Let $k \in \mathbb{N} \cup \{0\}$ and let $n = 2k+1$. Let Γ be as in the introduction, and let $\langle , \rangle^*$ be an A-invariant scalar product in $\mathbb{R}^n$. From definition (cf. [4] and [15]) $M = \mathbb{R}^n/\Gamma$ is a closed, orientable, flat manifold. Moreover the eigenvalues of the generator A of the holonomy group of M are equal to $e^{\frac{2\pi i j}{n}}, j = 1, ..., n$. In fact, for every $j = 2, ..., n-1$, consider the $(j \times j)$-matrix

$$M_j = \begin{bmatrix} 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \end{bmatrix}.$$

Let $M_j(z) = M_j - zI$. Then

$$\det(A - zI) = (1-z) \det M_{n-1}(z).$$

Applying the Laplace expansion with respect to the first row we have

$$\det M_j(z) = -z \det M_{j-1}(z) + (-1)^j.$$

Using this it is easy to check that $\det M_j(z) = (-1)^j \sum_{l=0}^{j} z^l$. Hence

$$\det(A - zI) = (1-z) \det M_{n-1}(z) = -z^n + 1.$$

Let $e_1, ..., e_n$ be an orthonormal basis in $(\mathbb{R}^n, \langle , \rangle^*)$. Throughout the rest of the paper we shall always assume (cf. [1, page 61] and [10, Proposition 1.3]) that:

(i) $e_1, ..., e_{n-1} \in \operatorname{Span}[a_1, ..., a_{n-1}]$ and $e_n = a_n$,
(ii) for every for $j \leq n-1$: $A(e_{2j-1}) = \cos(2\pi j/n) e_{2j-1} + \sin(2\pi j/n) e_{2j}$, and $A(e_{2j}) = -\sin(2\pi j/n) e_{2j-1} + \cos(2\pi j/n) e_{2j}$.

Let $\mathrm{Cliff}(n)$ be the Clifford algebra in $\mathbb{R}^n$ and let $\mathrm{Cliff}_{\mathbb{C}}(n)$ be its complexification. The group $Spin(n)$ is the set of products $x_1 \cdots x_{2r}$, where $r \in \mathbb{N}$, and where $x_1, ..., x_{2r}$ are the elements of the unit sphere in $\mathbb{R}^n$. The standard covering map $\lambda : Spin(n) \to SO(n)$ carries $y \in Spin(n)$ onto $\mathbb{R}^n \ni x \to yxy^*$,

where $(e_{j_1}\cdots e_{j_s})^* = e_{j_s}\cdots e_{j_1}$. A spin structure on an orientable flat manifold $M = \mathbb{R}^n/\Gamma$ is determined by the lift $\widehat{h} : \Gamma \to Spin(n)$ of the holonomy homomorphism $h : \Gamma \to SO(n)$. Recall that h carries $\gamma \in \Gamma$ onto its linear part $h(\gamma)$, (cf. [15, Chapter III]). For $M \in \mathcal{F}_{CHD}$ we have $h(\Gamma) = \langle A \rangle \cong \mathbb{Z}_n$ and any lift $\widehat{A}$ of A to $Spin(n)$ defines the lift $\widehat{h}$ of h, given by the formulas $\widehat{h}(a_j) = 1$ for $j \leq n-1$, $\widehat{h}(g) = \widehat{A}$. In order to construct $\widehat{A}$ consider $\beta = \frac{\pi}{n}$,

$$r_j = \cos(j\beta) + e_{2j-1}e_{2j}\sin(j\beta),$$

and $\alpha = \prod_{j=1}^{k} r_j$. Clearly $r_i r_j = r_j r_i$ for $i, j \in \{1, ..., k\}$. A direct calculation yields

$$\lambda(r_j)(e_l) = \begin{cases} \cos(2j\beta)e_{2j-1} + \sin(2j\beta)e_{2j} & \text{for } \ l = 2j-1 \\ -\sin(2j\beta)e_{2j-1} + \cos(2j\beta)e_{2j} & \text{for } \ l = 2j \\ e_l & \text{for } \ l \notin \{2j-1, 2j\} \end{cases}.$$

Using this it is easy to check that

$$\alpha^n = (-1)^{\frac{k(k+1)}{2}}$$

and $\lambda(\alpha) = A$. Now we can define

$$\alpha_+ = (-1)^{\frac{k(k+1)}{2}}\alpha, \quad \alpha_- = -(-1)^{\frac{k(k+1)}{2}}\alpha.$$

Since n is odd,

$$\alpha_+^n = 1 \quad \text{and} \quad \alpha_-^n = -1.$$

We have.

Lemma 1. *$H_1(M,\mathbb{Z}) \cong \mathbb{Z} \oplus H$, where H is a finite abelian group of odd order and $H^1(M,\mathbb{Z}_2) \cong \mathbb{Z}_2$.*

Proof: The group $\Gamma_0 = \langle a_1, ..., a_n \rangle$ is the maximal abelian subgroup of Γ and the following sequence

$$0 \to \Gamma_0 \to \Gamma \to \langle A \rangle \to 1$$

is exact (cf. [4, Proposition 4.1], [15, Theorem 3.2.9]). From [8, Corollary 1.3] we have $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes H_1(\Gamma, \mathbb{Z}) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes \Gamma_0^A) = 1$. Hence $H_1(M,\mathbb{Z}) \cong \mathbb{Z} \oplus H$, where H is a finite group. According to [3, Chapter 3], there are homomorphisms $res : H_*(M,\mathbb{Z}) \cong H_*(\Gamma,\mathbb{Z}) \to H_*(\Gamma_0,\mathbb{Z})$ and $cor : H_*(\Gamma_0,\mathbb{Z}) \to$

$H_*(M, \mathbb{Z})$ such that $cor \circ res$ is the multiplication by n. Since the group $H_*(\Gamma_0, Z) \cong H_*(T^n, \mathbb{Z})$ is torsion free we have $nH = 0$. In particular, the order of H is odd. For the proof of the last statment we have $H^1(M, \mathbb{Z}_2) \cong \mathrm{Hom}\,(H_1(M, \mathbb{Z}), \mathbb{Z}_2) \cong \mathrm{Hom}\,(\mathbb{Z}, \mathbb{Z}_2) \cong \mathbb{Z}_2$. $\square$

Since α_+, α_- are different lifts of the holonomy homomorphism h to $Spin(n)$, the spin structures determined by them are different. It is known that spin structures on M correspond to the elements of $H^1(M, \mathbb{Z}_2)$ ([7, p. 40]).

By [7, Section 1.3], the irreducible complex $\mathrm{Cliff}_{\mathbb{C}}(n)$-module Σ_{2k} can be described as follows. Consider

$$g_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

Let $\Sigma_{2k} = \underbrace{\mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2}_{k \text{ times}}$ and let $\alpha(j) = \begin{cases} 1 & \text{if } j \text{ is odd} \\ 2 & \text{if } j \text{ is even} \end{cases}$. Take an element $u = u_1 \otimes ... \otimes u_k$ of Σ_{2k} and the orthonormal basis $e_1, .., e_n$ considered above. Then

$$e_j u = (I \otimes ... \otimes I \otimes g_{\alpha(j)} \otimes \underbrace{T \otimes ... \otimes T}_{[\frac{j-1}{2}] \text{ times}})(u),$$

for $j \leq n-1$, and

$$e_n u = i(T \otimes ... \otimes T)u.$$

A spin structure on M determines a complex spinor bundle $P\Sigma_{2k}$ with fiber Σ_{2k}. This bundle is the orbit space of $\mathbf{R}^n \times \Sigma_{2k}$ by the action of Γ given by

$$\gamma(x, v) = (\gamma x, \widehat{h}(\gamma)v), \tag{1}$$

where $\gamma \in \Gamma, x \in \mathbf{R}^n$ and $v \in \Sigma_{2k}$. Clearly

$$\widehat{h}(a_j) = 1 \text{ for } j \leq n-1$$

and

$$\widehat{h}(g) = \alpha_{\pm}.$$

Since $\mathrm{Span}[e_1, ..., e_{n-1}] = \mathrm{Span}[a_1, ..., a_{n-1}]$ and $a_n = e_n$ we conclude that

$$\widehat{h}(e_j) = 1 \text{ for } j \leq n-1.$$

Consider the covering $T^n = \mathbb{R}^n/\Gamma_0 \to M$. We have $\widehat{h}(a_n) = \pm 1$. The lift $P_T\Sigma_{2k}$ of $P\Sigma_{2k}$ to T^n is the orbit space $(\mathbb{R}^n \times \Sigma_{2k})/\Gamma_0$, where the action of Γ_0 on $\mathbb{R}^n \times \Sigma_{2k}$ is given by the formula (1).

To deal with the spectrum of the Dirac operator D it is convenient to describe it in terms of the spectrum of D^2. We state without proofs some related results of [12] that will be used later. Identify the parallel section $\mathbb{R}^n \ni x \to (x, v) \in \mathbb{R}^n \times \Sigma_{2k}$ with v. Every section (spinor) of the trivial bundle $\mathbb{R}^n \times \Sigma_{2k}$ (covering our bundle $P\Sigma_{2k}$) can be written as a linear combination of fv, where $f \in C^\infty(\mathbb{R}^n, \mathbb{C})$ and v is a parallel section. Take the coordinate system $x_1, ..., x_n$ determined by $e_1, ..., e_n$. Since v is parallel,

$$D(fv) = \sum_j e_j \nabla_{e_j}(fv) = \sum_j e_j \left(\tfrac{\partial}{\partial x_j}(f)v + f\nabla_{e_j} v \right) = \sum_j e_j \tfrac{\partial f}{\partial x_j} v. \quad (2)$$

Let Γ_0^* be the dual lattice of Γ_0. Let $\mathcal{B}$ be Γ_0^* in the case of α_+ and $\Gamma_0^* + \frac{1}{2}e_n$ in the case of α_-. The action of g on on the set of sections of $\mathbb{R}^n \times \Sigma_{2k}$, induced by the action of g on $\mathbb{R}^n$, is given by the formula

$$g(\phi)(x) = \widehat{h}(g)\phi(g^{-1}x), \quad (3)$$

where ϕ is a spinor on $\mathbb{R}^n$.

Consider $f_b(x) = e^{2\pi i \langle b, x\rangle}$. By immediate calculation or following ([12]) we have

$$D^2(f_b v) = 4\pi^2 ||b||^2 f_b v. \quad (4)$$

Hence the sections $f_b v, b \in \mathcal{B}$, $v \in \Sigma_{2k}$, correspond to eigenvectors of D on T^n, and the elements of $\{f_b v : v \in \Sigma_{2k}, b \in \mathcal{B}\}^g$ correspond to eigenvectors of D on M.

For $b \in \mathcal{B}$, let us denote the corresponding D^2-eigenspace by $E_b(D^2) = \mathrm{Span}\{f_b v : v \in \Sigma_{2k}\}$. We have the decomposition $E_b(D^2) = E_{b+}(D) \oplus E_{b-}(D)$, where

$$E_{b\pm}(D) = \{p \in E_b(D^2) : \, Dp = \pm 2\pi ||b|| \, p\}.$$

Since

$$(f_b \circ g^{-1})(x) = e^{-2\pi i \langle A(b), a\rangle} f_{A(b)}(x) \quad (5)$$

we have $AE_b \subseteq E_{A(b)}$, (cf. [12, Lemma 4.1]). Denote $< A >$ by G.

Let $\mathcal{B}_{Sym} = \{b \in \mathcal{B} : \#G(b) = \#G\}$, $\mathcal{B}_{Pas} = \{b \in \mathcal{B} : \#G(b) < \#G\}$ and

$$D_S = D|_{[\bigoplus_{b \in \mathcal{B}_{Sym}} E_b(D^2)]^g}, \quad D_{Pas} = D|_{[\bigoplus_{b \in \mathcal{B}_{Pas}} E_b(D^2)]^g}. \quad (6)$$

Clearly $\mathcal{B}$ is the disjoint union of $\mathcal{B}_{Sym}$ and $\mathcal{B}_{Pas}$ and the Dirac operator D on M can be identified with $D_S \oplus D_{Pas}$. If $b \in \mathcal{B}_{Sym}$ and

$$V_b^{\pm} = \bigoplus_{h \in G} E_{h(b\pm)}(D^2)$$

then $\dim(V_b^{\pm})^g = \dim E_b^{\pm}(D) = 2^{k-1}$ (cf. [12, Theorem 4.2, Corollary 4.3]).

3 Eta invariants of $\mathcal{F}_{CHD}$-manifolds

The aim of this section is to prove Theorem 1. Recall that the η-invariant of the Dirac operator on a closed spin manifold M is defined as follows. As D is elliptic formally self adjoint, it has discrete real spectrum and the series $\sum_{\lambda \neq 0} \operatorname{sgn} \lambda |\lambda|^{-z}$ converges for $z \in \mathbb{C}$ with $\operatorname{Re}(z)$ sufficiently large ([1, Theorem 3.10]). Here summation is taken over all nonzero eigenvalues λ of D, each eigenvalue being repeated according to its multiplicity. The function $z \to \sum_{\lambda \neq 0} \operatorname{sgn} \lambda |\lambda|^{-z}$ can be extended to a meromorphic function η_M in the whole complex plane such that 0 is a regular point of η_M ([1, Theorem 3.10]). The *eta-invariant* of M is $\eta_M(0)$.

Define an endomorphism ρ_1 of $\mathbb{C}^2$ by the formula

$$\rho_1(u) = \cos\beta\, u + \sin\beta\, g_1 g_2\, u.$$

The matrix of ρ_1 is equal to

$$\cos\beta\, I + \sin\beta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

so that the matrix of ρ_1^j is equal to

$$\cos(\beta j)\, I + \sin(\beta j) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The following lemma is crucial.

Lemma 2. *Let $w_{+1} = (1, -i)$, let $w_{-1} = (1, i)$, let $\epsilon = (\epsilon_1, ..., \epsilon_k) \in \{-1, 1\}^k$, and let $v_\epsilon = w_{\epsilon_1} \otimes ... \otimes w_{\epsilon_k}$. Let $\beta = \frac{\pi}{n}$ and let μ_ϵ be as in Theorem 1. Take $u = u_1 \otimes .... \otimes u_k \in \Sigma_{2k}$. Then*

a) $\alpha u = \rho_1 u_1 \otimes .... \otimes \rho_1^k u_k$,

b) $\alpha e_n u = e_n \alpha u$,

c) $\rho_1(w_{\pm 1}) = e^{\pm i\beta} w_{\pm 1}$,

d) $\alpha v_\epsilon = e^{i\beta\mu_\epsilon} v_\epsilon$ *and* $\{v_\epsilon : \epsilon \in \{-1,1\}^k\}$ *is a basis of* Σ_{2k},

e) $e_n v_\epsilon = -i\nu(\epsilon) v_\epsilon$.

Proof. a) Since $T^2 = id$,

$$e_{2j-1}e_{2j}(u_1 \otimes ... \otimes u_j \otimes ... \otimes u_k) = u_1 \otimes ... \otimes g_1 g_2(u_j) \otimes ... \otimes u_k$$

and consequently

$$r_j(u_1 \otimes ... \otimes u_j \otimes ... \otimes u_k) = u_1 \otimes ... \otimes \rho_1^j(u_j) \otimes ... \otimes u_k.$$

Hence

$$\alpha(u_1 \otimes ... \otimes u_k) = (r_1 \cdots r_k)(u_1 \otimes ... \otimes u_k) = \rho_1(u_1) \otimes ... \otimes \rho_1^k(u_k).$$

b) For $j \le k$ we have $e_{2j-1}e_{2j}e_n = e_n e_{2j-1}e_{2j}$ so that $r_j e_n = e_n r_j$.

c) is obvious.

d) By c), $r_1(w_{\epsilon_j}) = e^{i\beta\epsilon_j} w_{\epsilon_j}$. Hence

$$\alpha(v_\epsilon) = r_1(w_{\epsilon_1}) \otimes ... \otimes r_1^k(w_{\epsilon_k}) = e^{i\beta\mu_\epsilon} v_\epsilon.$$

Since $\#\{v_\epsilon : \epsilon \in \{-1,1\}^k\} = 2^k = \dim \Sigma_{2k}$ and the vectors v_ϵ are linearly independent, they form a basis of Σ_{2k}.

e) We have $T(w_1) = -w_1$ and $T(w_{-1}) = w_{-1}$. It follows that

$$e_n v_\epsilon = iTw_{\epsilon_1} \otimes ... \otimes Tw_{\epsilon_k} = i(-1)^{\#\{j \in \{1,..,k\} : \epsilon_j = 1\}} v_\epsilon = -i\nu(\epsilon) v_\epsilon.$$

This finishes the proof of Lemma 2. $\square$

Let $\mathcal{E}(\lambda, D_{Pas})$ be the eigenspace of λ for D_{Pas} on M. From the definition and (4), (5) it is easy to see that $\lambda = 2\pi l$ in the case of α_+ and $\lambda = 2\pi(l + \frac{1}{2})$ in the case of α_-, where $l \in \mathbb{Z}$. In the case α_+, $\mathcal{B}_{Pas} = \{le_n : l \in \mathbb{Z}\}$ and we have

$$D(f_{le_n} v_\epsilon) = -\tfrac{\partial}{\partial x_n}(e^{2\pi i \langle le_n, x\rangle}) i\nu(\epsilon) v_\epsilon = \nu(\epsilon) 2\pi l f_{le_n} v_\epsilon. \tag{7}$$

Hence $\mathcal{E}(2\pi l, D_{Pas}) = \mathrm{Span}[f_{\nu(\epsilon)le_n} v_\epsilon : \epsilon \in \{-1,1\}^k]^g$. Similar formulas are also true for α_-, where $\mathcal{B}_{Pas} = \{(l + \frac{1}{2})e_n : l \in \mathbb{Z}\}$.

Now we are able to describe the spectrum of D_{Pas}.

Proposition 2. *Let n, k, M, μ_ϵ, $\nu(\epsilon)$, and $c(k)$ be as in Theorem 1. Let $b^+ = le_n$ and $b^- = (l+\frac{1}{2})e_n$, where $l \in \mathbb{Z}$.*

a) *If the spin structure is given by α_+, then*

$$\mathcal{E}(2\pi l, D_{Pas}) = \mathrm{Span}[f_{\nu(\epsilon)b^+}v_\epsilon : \frac{\mu_\epsilon}{2} + c(k)n \equiv \nu(\epsilon)l \mod (n)].$$

b) *If the spin structure is given by α_-, then*

$$\mathcal{E}(2\pi(l+\frac{1}{2}), D_{Pas}) = \mathrm{Span}[f_{\nu(\epsilon)b^-}v_\epsilon : \frac{\mu_\epsilon}{2} + c(k)n + \frac{n-1}{2} \equiv \nu(\epsilon)l \mod (n)].$$

Remark 1. Since $\epsilon_j - 1$ are even, the difference $\mu_\epsilon - k(k+1)/2 = \sum_{j=1}^k \epsilon_j j - \sum_{j=1}^k j$ is divisible by 2. Using this and the definition of $c(k)$ it is easy to see that $\mu_\epsilon/2 + c(k)n$ is an integer.

Proof of Proposition 2. a) From the definitions of α_+ and $c(k)$ it follows that $\alpha_+ = (-1)^{2c(k)}\alpha$. We have

$$g(f_{le_n}(x)v_\epsilon) = f_{le_n}(g^{-1}x)\alpha_+v_\epsilon = e^{-\frac{2\pi il}{n}}f_{le_n}(x)(-1)^{2c(k)}\alpha v_\epsilon$$
$$= e^{-\frac{2\pi i}{n}(l-\frac{\mu_\epsilon}{2}-c(k)n)}f_{le_n}(x)v_\epsilon.$$

By the above one gets the required conditions.

b) In the case of α_- the eigenvectors of D_{Pas} on T^n can be written as $f_{(l+\frac{1}{2})e_n}v$ for $l \in \mathbb{Z}, v \in \Sigma_{2k}$. We have

$$gf_{(l+\frac{1}{2})e_n}(x)v_\epsilon = e^{-\frac{2\pi i}{n}(l-c(k)n-\frac{n-1}{2}-\frac{\mu_\epsilon}{2})}f_{(l+\frac{1}{2})e_n}(x)v_\epsilon.$$

Hence $f_{(b^-)}v_\epsilon$ is g-equivariant if and only if $l \in n\mathbb{Z} + c(k)n + \frac{n-1}{2} + \frac{\mu_\epsilon}{2}$. The rest of the argument is the same as in a). This finishes the proof of Proposition 2. □

Lemma 3. *Let $M \in \mathcal{F}_{CHD}$ be an n-dimensional with $k = [\frac{n-1}{2}]$ odd and with a fixed spin structure. Let m be a natural number such that $m \equiv r \mod (n)$. Assume that $f_bv_\epsilon \in \mathcal{E}(\lambda, D_{Pas})$. Then*

a) $f_{-b}v_{-\epsilon} \in \mathcal{E}(\lambda, D_{Pas})$,

b) $\dim \mathcal{E}(2\pi(m), D_{Pas}) = A_r^+$ *and* $\dim \mathcal{E}(2\pi(m+\frac{1}{2}), D_{Pas}) = A_r^-$.

Proof. **a)** If the spin structure on M is α_+, then $b = le_n$ for some $l \in \mathbb{Z}$, and, from the equivariance of $f_b v_\epsilon$, it follows that

$$l \equiv \tfrac{\mu_\epsilon}{2} + c(k)n \bmod (n).$$

Hence

$$-l \equiv \tfrac{\mu_{-\epsilon}}{2} + c(k)n \bmod (n)$$

and $f_{-b}v_{-\epsilon}$ is g-equivariant. By the assumption that k is odd, $\nu(-\epsilon) = -\nu(\epsilon)$. According to Lemma , $f_{-b}v_{-\epsilon} \in \mathcal{E}(\lambda, D_{Pas})$.

If the spin structure is α_-, then we use the congruence

$$l \equiv \tfrac{\mu_\epsilon}{2} + c(k)n + k \bmod (n).$$

Since $\mu_{-\epsilon} = -\mu_\epsilon$, $c(k)n \equiv -c(k) \bmod (n)$, and $-k-1 \equiv k \bmod (n)$ we have

$$-l - 1 \equiv \tfrac{\mu_{-\epsilon}}{2} + c(k)n + k \bmod (n)$$

and consequently $f_{-b}v_{-\epsilon}$ is g-equivariant.

b) If the spin structure is α_+, then using Proposition 2 and a), we get

$$\dim \mathcal{E}(2\pi(m), D_{Pas}) = \dim \mathcal{E}(2\pi(r), D_{Pas})$$

$$= 2\#\{\epsilon \in \mathcal{D}_+ : \tfrac{\mu_\epsilon}{2} + c(k)n \equiv r \bmod (n)\} = A_r^+.$$

In the case of α_- we get

$$\dim \mathcal{E}(2\pi(m + \frac{1}{2}), D_{Pas}) = \dim \mathcal{E}(2\pi(r + \frac{1}{2}), D_{Pas})$$

$$= 2\#\{\epsilon \in \mathcal{D}_+ : \frac{\mu_\epsilon}{2} + c(k)n + k \equiv r \mod (n)\} = A_r^-.$$

$\square$

Proof of Theorem 1. We shall modify of a proof of Lemma 5.5 from [12].

a) Let $m \in \mathbb{Z}$, and let $\mathcal{S}_r = \{2\pi(m) : m \equiv r \bmod (n)\}$. It is clear that $\mathcal{S}_r$ are disjoint and $\mathcal{S}_{Pas} \subseteq \Cup_{r=0}^{n-1} \mathcal{S}_r$. Since D_S has symmetric spectrum,

$$\eta_M(z) = \sum_{\lambda \in \mathcal{S}_{Pas}} \frac{\operatorname{sgn}(\lambda)}{|\lambda|^z} \dim \mathcal{E}(\lambda, D_{Pas})$$

for Re(z) sufficiently large. By Lemma 3 b), $\dim \mathcal{E}(\lambda, D_{Pas}) = A_r^+$ for $\lambda \in \mathcal{S}_r$. If $A_0^+ \neq 0$, then the eigenvalue $\lambda \in \mathcal{S}_0$ occur together with $-\lambda$ with the same multiplicity A_0^+ so that $\sum_{\lambda \in \mathcal{S}_0 - \{0\}} \frac{A_0^+ \operatorname{sgn}(\lambda)}{|\lambda|^z} = 0$ and, for Re(z) sufficiently big,

$$\eta_M(z) = \sum_{r=1}^{n-1} \sum_{m=-\infty}^{\infty} \frac{A_r^+ \operatorname{sgn}(2\pi(mn+r))}{|2\pi(mn+r)|^z} = \sum_{r=1}^{n-1} \sum_{m=-\infty}^{\infty} \frac{A_r^+ \operatorname{sgn}\left(2\pi n\left(m+\frac{r}{n}\right)\right)}{\left|2\pi n\left(m+\frac{r}{n}\right)\right|^z}$$

$$= \sum_{r=1}^{n-1} \frac{A_r^+}{|\ 2\pi n\ |^z} \left(\sum_{m=0}^{\infty} \frac{1}{(m+\frac{r}{n})^z} - \sum_{m=0}^{\infty} \frac{1}{(m+1-\frac{r}{n})^z} \right).$$

The last two series are known as generalized zeta functions (cf. [14]). They have meromorphic extensions on $\mathbb{C}$ without poles in $z = 0$. Let $\zeta(z, a)$ denote the function defined by $\sum_{m=0}^{\infty} \frac{1}{(m+\frac{r}{n})^z}$ for Re(z) sufficiently big. One gets for the extension: $\zeta(0, a) = \frac{1}{2} - \frac{r}{n}$. Hence

$$\eta_M(0) = \sum_{r=1}^{n-1} A_r^+ \left(1 - \frac{2r}{n}\right).$$

b) We use similar arguments as those given in the proof of a). Now the component $\mathcal{S}_0$ is not symmetric so that we do not remove $r = 0$ from the formula describing $\eta_M(z)$. The equality

$$2\pi\left(mn + r + \frac{1}{2}\right) = 2\pi n\left(m + \frac{2r+1}{2n}\right)$$

and the above considerations implies that

$$\eta_M(0) = \sum_{r=0}^{n-1} A_r^- \left(1 - \frac{2r+1}{n}\right).$$

This finishes the proof of Theorem 1. □

We have.

Corollary 1 *Let n be a prime number greater than 3 such that $n+1$ is divisible by 4 and let $M^n \in \mathcal{F}_{CHD}$ be a flat manifold with a fixed spin structure. Then $\eta_{M^n} \in \mathbb{Z}$.*

Proof: Let $l = \frac{n+1}{4}$. It is known (cf. [13, chapter 9] that 2^s copies of M^n is a boundary of a spin manifold W^{n+1} for some $s \in \mathbb{N}$. By [1, Theorem 4.2]

$$\int_{W^{n+1}} \hat{A}_l(p) - \frac{2^s \eta_{M^n}}{2} \in \mathbb{Z},$$

where $\hat{A}_l$ is the l-th $\hat{A}$-polynomial on Pontriagin classes. By [9] $\int_{W^{n+1}} \hat{A}_l(p)$ can be written as $\frac{C_{W^{n+1}}}{q_1 \ldots q_r}$, where $C_{W^{n+1}} \in \mathbb{Z}$ and where $q_1, \ldots, q_r \in \{2, 3, ..., n-1\}$ are prime numbers. From Theorem 1 $\eta_{M^n} = \frac{C_{M^n}}{n}$, for some $C_{M^n} \in \mathbb{Z}$. Since $\frac{C_{W^{n+1}}}{q_1 \ldots q_r} - 2^{s-1}\eta_{M^n} \in \mathbb{Z}$ we have $\frac{2^{s-1} q_1 \ldots q_r C_{M^n}}{n} \in \mathbb{Z}$. Hence $\eta_{M^n} \in \mathbb{Z}$. $\square$

Corollary 2 *Let n be an odd number. If $M^n \in \mathcal{F}_{CHD}$, then $d = \eta_{M^n,\alpha^+} - \eta_{M^n,\alpha^-} \in 2\mathbb{Z}$.*

Proof: By [5, Theorem 1.1], $d \in \frac{1}{2}\mathbb{Z}$. From the definition (cf. page 2) all $A_r^{\pm}$ belongs to $2\mathbb{Z}$. Hence $d = \frac{2C}{n}$ for some $C \in \mathbb{Z}$. Summing up $\frac{dn}{2} \in \mathbb{Z}$ and $d \in 2\mathbb{Z}$. $\square$

Example 1. We calculate η_{M^7,α_+}, where$M^7 \in \mathcal{F}_{CHD}$. Since $\frac{k(k+1)}{2} = 6$ is even our equation is

$$r \equiv \frac{\mu_\epsilon}{2} \bmod (7).$$

The values of $\frac{\mu_\epsilon}{2}$ and r for $\epsilon \in \mathcal{D}_+$ are given in the following table.

ϵ	$\frac{\mu_\epsilon}{2}$	r
$(1,1,1)$	3	3
$(1,-1,-1)$	-2	5
$(-1,1,-1)$	-1	6
$(-1,-1,1)$	0	0

It follows that $A_j^+ = 2$, for $j = 0, 3, 5, 6$ and $A_j^+ = 0$ for other values of j. By Theorem 1,

$$\eta_M^7(0) = 2\left[\left(1 - \frac{6}{7}\right) + \left(1 - \frac{10}{7}\right) + \left(1 - \frac{12}{7}\right)\right] = -2.$$

4 Harmonic spinors on $\mathcal{F}_{CHD}$-manifolds

A harmonic spinor on a closed spin manifold M is an element of the kernel of the Dirac operator on M.

Proof of Proposition 1. a) We have

$$gv_\epsilon = (-1)^{\frac{k(k+1)}{2}}\alpha v_\epsilon = (-1)^{\frac{k(k+1)}{2}} e^{\frac{2\pi i}{2n}\mu_\epsilon} v_\epsilon.$$

First consider the case when $k(k+1)/2$ is even. Then $gv_\epsilon = v_\epsilon$ if and only if

$$\mu_\epsilon \equiv 0 \bmod (2n)$$

and $k = 4k_0 + 3$ or $k = 4k_0$. Let δ_4 denote the sequence $1, -1, -1, 1$. If $k = 4k_0 + 3$ and

$$\epsilon = (-1, -1, 1, \underbrace{\delta_4, ..., \delta_4}_{k_0 \text{ times}}),$$

then ϵ belongs to $\{-1, 1\}^k$ and $\mu_\epsilon = 0$. If $k = 4k_0$ and

$$\epsilon = (\underbrace{\delta_4, ..., \delta_4}_{k_0 \text{ times}}),$$

then ϵ belongs to $\{-1, 1\}^k$ and $\mu_\epsilon = 0$. In particular, $\mathfrak{h}(M) > 0$.

Now assume that $k > 2$ and $k(k+1)/2$ is odd. Then $gv_\epsilon = v_\epsilon$ if and only if

$$\mu_\epsilon \equiv n \bmod (2n)$$

and $k = 4k_0 + 1$ or $k = 4k_0 + 2$. If $k = 4k_0 + 1$ consider

$$\epsilon = (1, -1, 1, \underbrace{\delta_4, ..., \delta_4}_{k_0 - 1 \text{ times}}, 1, 1)$$

and if $k = 4k_0 + 2$ consider

$$\epsilon = (-1, 1 - 1, 1, \underbrace{\delta_4, ..., \delta_4}_{k_0 - 1 \text{ times}}, 1, 1).$$

In both cases $\epsilon \in \{-1, 1\}^k$ and $\mu_\epsilon = n$. It is easily seen that the equation $\mu_\epsilon \equiv 0 \bmod (2n)$ have no solutions for $k = 1$ or 2.

b) Since $g^n = -id$, the equation $gv = v$ have only one solution $v = 0$. $\square$

It is easy to see that the equality $\alpha_+ v_\epsilon = v_\epsilon$ implies $\alpha_+ v_{-\epsilon} = v_{-\epsilon}$. Using this and the arguments given in the proof of Proposition 1 we have.

Corollary 3 *If $M \in \mathcal{F}_{CHD}$ and* $\dim M = 2k+1$, *then*

$$\mathfrak{h}(M, \alpha_+) = 2\#\{\epsilon \in \mathcal{D}_+ : \frac{\mu_\epsilon}{2} + c(k)n \equiv 0 \bmod (2k+1)\}.$$

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